

HOCHSCHILD COHOMOLOGY OF TRIANGULAR MATRIX ALGEBRAS

JORGE A. GUCCIONE AND JUAN J. GUCCIONE

ABSTRACT. Let $E = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be a triangular algebra, where A and B are algebras over an arbitrary commutative ring k and M is an (A, B) -bimodule. We prove the existence of two long exact sequence of k -modules relating the Hochschild cohomology of A , B and E . We also study the structure of the maps of the first of these exact sequences.

INTRODUCTION

Let k be an arbitrary commutative ring with unit, A and B two k -algebras with unit, M an (A, B) -bimodule, $E = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ the triangular algebra and X an E -bimodule. Let 1_A and 1_B be the unit elements of A and B respectively. Let us write $X_{AA} = 1_A X 1_A$, $X_{AB} = 1_A X 1_B$, $X_{BA} = 1_B X 1_A$ and $X_{BB} = 1_B X 1_B$. For example, for $X = E$, we have $X_{AA} = A$, $X_{AB} = M$, $X_{BA} = 0$ and $X_{BB} = B$.

The purpose of this paper is to prove the following results:

Theorem 1. *There exists a long exact sequence*

$$0 \rightarrow H^0(E, X) \xrightarrow{j^0} H^0(A, X_{AA}) \oplus H^0(B, X_{BB}) \xrightarrow{\delta^0} \text{Ext}_{A \otimes B^{\text{op}}, k}^0(M, X_{AB}) \rightarrow \\ \xrightarrow{\pi^0} H^1(E, X) \xrightarrow{j^1} H^1(A, X_{AA}) \oplus H^1(B, X_{BB}) \xrightarrow{\delta^1} \text{Ext}_{A \otimes B^{\text{op}}, k}^1(M, X_{AB}) \rightarrow \dots,$$

where $\text{Ext}_{A \otimes B^{\text{op}}, k}^*(M, X_{AB})$ denotes the Ext groups of the $A \otimes B^{\text{op}}$ -module M , relative to the family of the $A \otimes B^{\text{op}}$ -linear epimorphisms which split as k -linear morphisms.

Let $\pi: E \rightarrow B$ the ring morphism defined by $\pi\left(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\right) = b$. We let B_E denote the ring B consider as an E -bimodule via π .

Theorem 2. *There exists a long exact sequence*

$$0 \rightarrow \text{Ext}_{E \otimes B^{\text{op}}, k}^0(B_E, X 1_B) \rightarrow H^0(E, X) \rightarrow H^0(A, X_{AA}) \rightarrow \\ \rightarrow \text{Ext}_{E \otimes B^{\text{op}}, k}^1(B_E, X 1_B) \rightarrow H^1(E, X) \rightarrow H^1(A, X_{AA}) \rightarrow \dots,$$

2000 *Mathematics Subject Classification.* Primary 16E40; Secondary 16S50.
Supported by UBACYT 01/TW79 and CONICET

where $\text{Ext}_{E \otimes B^{\text{op}}, k}^*(B_E, X1_B)$ denotes the Ext groups of the $E \otimes B^{\text{op}}$ -module B_E , relative to the family of the $E \otimes B^{\text{op}}$ -linear epimorphisms which split as k -linear morphisms.

Let C be a k -algebra. The complex $\text{Hom}_{C^e}((C^{*+2}, b'_*), C)$ is a differential graded algebra via the cup product

$$(fg)(x_0 \otimes \cdots \otimes x_{r+s+1}) = f(x_0 \otimes \cdots \otimes x_r \otimes 1_E)g(1_E \otimes x_{r+1} \otimes \cdots \otimes x_{r+s+1}),$$

where $f \in \text{Hom}_{C^e}(C^{r+2}, C)$ and $g \in \text{Hom}_{C^e}(C^{s+2}, C)$. Hence $H^*(E, E)$ becomes a graded associative algebra. As it is well known, $H^*(E, E)$ is a graded commutative algebra. Similarly, for a left C -module Y , the complex $\text{Hom}_C((C^{*+1} \otimes Y, b'_*), Y)$ is a differential graded algebra via

$$(fg)(x_0 \otimes \cdots \otimes x_{r+s} \otimes y) = f(x_0 \otimes \cdots \otimes x_r \otimes g(1 \otimes x_{r+1} \otimes \cdots \otimes x_{r+s} \otimes y)),$$

where $f \in \text{Hom}_C(C^{r+1} \otimes Y, Y)$ and $g \in \text{Hom}_C(C^{s+1} \otimes Y, Y)$. Hence $\text{Ext}_{C, k}^*(Y, Y)$ is a graded associative algebra. In the following theorem we consider $H^*(A, A)$, $H^*(B, B)$, $H^*(E, E)$ and $\text{Ext}_{A \otimes B^{\text{op}}, k}^*(M, M)$ equipped with these algebra structures.

Theorem 3. *Assume that $X = E$. Then*

- (1) *The map $H^*(E, E) \xrightarrow{j^*} H^*(A, A) \oplus H^*(B, B)$ is a morphism of graded rings.*
- (2) *The maps $H^*(A, A) \hookrightarrow H^*(A, A) \oplus H^*(B, B) \xrightarrow{\delta^*} \text{Ext}_{A \otimes B^{\text{op}}, k}^*(M, M)$ and $H^*(B, B) \hookrightarrow H^*(A, A) \oplus H^*(B, B) \xrightarrow{\delta^*} \text{Ext}_{A \otimes B^{\text{op}}, k}^*(M, M)$ are morphisms of graded rings.*
- (3) *The map $\text{Ext}_{A \otimes B^{\text{op}}, k}^*(M, M) \xrightarrow{\pi^*} H^{*+1}(E, E)$ has image in the annihilator of $\bigoplus_{n \geq 1} H^n(E, E)$.*

Theorems 1 and 2, which generalize a previous result of [H], were established in [M-P] under the assumptions that k is a field, A and B are finite dimensional k -algebras, M is a finitely generated (A, B) -bimodule and $X = E$. As was pointed out in [M-P], this version of Theorem 1, also follows from a result of [C]. Theorem 3 was proved in [G-M-S, Setion 5], under the assumptions that k is a field, A is a finite dimensional k -algebra and E is a one point extension of A .

Our proofs are elementary. The main tool that we use is the existence of a simple relative projective resolution of E .

Next, we enunciate the homological versions of Theorems 1 and 2. Similar methods to the ones used to prove Theorems 1 and 2 work in the homological context. We left the task of giving the proofs to the reader.

Theorem 1'. *Let $X_{BA} = 1_B X 1_A$. There exists a long exact sequence*

$$\begin{aligned} \cdots \rightarrow \text{Tor}_1^{A \otimes B^{\text{op}}, k}(M, X_{BA}) \rightarrow H_1(A, X_{AA}) \oplus H_1(B, X_{BB}) \rightarrow H_1(E, X) \rightarrow \\ \rightarrow \text{Tor}_0^{A \otimes B^{\text{op}}, k}(M, X_{BA}) \rightarrow H_0(A, X_{AA}) \oplus H_0(B, X_{BB}) \rightarrow H_0(E, X) \rightarrow 0, \end{aligned}$$

where $\text{Tor}_*^{A \otimes B^{\text{op}}, k}(M, X_{BA})$ denotes the Tor groups of the $A \otimes B^{\text{op}}$ -module M , relative to the family of the $A \otimes B^{\text{op}}$ -linear epimorphisms which split as k -linear morphisms.

Theorem 2'. *There exists a long exact sequence*

$$\begin{aligned} \dots \rightarrow H_1(A, X_{AA}) \rightarrow H_1(E, X) \rightarrow \operatorname{Tor}_1^{E \otimes B^{\text{op}}, k}(B_E, 1_B X) \rightarrow \\ \rightarrow H_0(A, X_{AA}) \rightarrow H_0(E, X) \rightarrow \operatorname{Tor}_0^{E \otimes B^{\text{op}}, k}(B_E, 1_B X) \rightarrow 0, \end{aligned}$$

where $\operatorname{Tor}_*^{E \otimes B^{\text{op}}, k}(B_E, 1_B X)$ denotes the Tor groups of the $E \otimes B^{\text{op}}$ -module B_E , relative to the family of the $E \otimes B^{\text{op}}$ -linear epimorphisms which split as k -linear morphisms.

Remark. When the present paper was finished we learned that Theorem 1 was also obtained in [C-M-R-S] under the additional assumptions that k is a field, $X = E$ and M is A -projective on the left or B -projective on the right.

PROOF OF THE RESULTS

Let (E^{*+2}, b'_*) be the canonical resolution of E and let (X_*, b'_*) be the E -bimodule subcomplex of (E^{*+2}, b'_*) , defined by

$$X_n = A^{n+2} \oplus B^{n+2} \oplus \bigoplus_{i=0}^{n+1} A^i \otimes M \otimes B^{n+1-i}.$$

It is easy to see that (X_*, b'_*) is a direct summand of (E^{*+2}, b'_*) as an E -bimodule complex. Moreover, the complex

$$E \xleftarrow{b'_0} X_0 \xleftarrow{b'_1} X_1 \xleftarrow{b'_2} X_2 \xleftarrow{b'_3} X_3 \xleftarrow{b'_4} X_4 \xleftarrow{b'_5} X_5 \xleftarrow{b'_6} X_6 \xleftarrow{b'_7} X_7 \xleftarrow{b'_8} \dots$$

is contractible as a right E -module complex. Hence, (X_*, b'_*) is a projective resolution of the E^e -module E , relative to the family of the E^e -linear epimorphisms which split as k -linear morphisms.

Let (X_*^A, b'_*) and (X_*^B, b'_*) be the subcomplexes of (X_*, b'_*) , defined by $X_n^A = A^{n+1} \otimes (A \oplus M)$ and $X_n^B = (B \oplus M) \otimes B^{n+1}$. It is easy to see that (X_*^A, b'_*) and (X_*^B, b'_*) are projective resolutions of the E^e -modules $1_A E$ and $E 1_B$ respectively, relative to the family of the E^e -linear epimorphisms which split as k -linear morphisms. We have the following:

Lemma 3. *Let $\mu: \frac{X_1}{X_1^A \oplus X_1^B} \rightarrow M$ be the map defined by $\mu(a \otimes m \otimes b) = amb$, for $a \in A$, $b \in B$ and $m \in M$. The complex*

$$(*) \quad M \xleftarrow{\mu} \frac{X_1}{X_1^A \oplus X_1^B} \xleftarrow{b'_2} \frac{X_2}{X_2^A \oplus X_2^B} \xleftarrow{b'_3} \frac{X_3}{X_3^A \oplus X_3^B} \xleftarrow{b'_4} \frac{X_4}{X_4^A \oplus X_4^B} \xleftarrow{b'_5} \dots,$$

is a relative projective resolution of M as an E -bimodule. A contracting homotopy of $(*)$ as a complex of k -modules is the family $\sigma_1: M \rightarrow \frac{X_1}{X_1^A \oplus X_1^B}$ and

$\sigma_{n+1}: \frac{X_n}{X_n^A \oplus X_n^B} \rightarrow \frac{X_{n+1}}{X_{n+1}^A \oplus X_{n+1}^B}$ ($n \geq 1$), defined by:

$$\sigma_1(m) = 1_A \otimes m \otimes 1_B,$$

$$\sigma_{n+1}(a_0 \otimes m \otimes \mathbf{b}_{2,n+1}) = 1_A \otimes a_0 \otimes m \otimes \mathbf{b}_{2,n+1} + (-1)^n 1_A \otimes a_0 m \otimes \mathbf{b}_{2,n+1} \otimes 1_B,$$

$$\sigma_{n+1}(\mathbf{a}_{0,i} \otimes m \otimes \mathbf{b}_{i+2,n+1}) = 1_A \otimes \mathbf{a}_{0,i} \otimes m \otimes \mathbf{b}_{i+2,n+1} \quad \text{for } i > 0,$$

where $\mathbf{a}_{0i} = a_0 \otimes \dots \otimes a_i$ and $\mathbf{b}_{i+2,n+1} = b_{i+2} \otimes \dots \otimes b_{n+1}$.

Proof. It follows by a direct computation. \square

Lemma 4. *We have*

$$\begin{aligned}\mathrm{Hom}_{A^e}((A^{*+2}, b'_*), X_{AA}) &\simeq \mathrm{Hom}_{E^e}((X_*^A, b'_*), X), \\ \mathrm{Hom}_{B^e}((B^{*+2}, b'_*), X_{BB}) &\simeq \mathrm{Hom}_{E^e}((X_*^B, b'_*), X).\end{aligned}$$

Proof. Since, for every $f \in \mathrm{Hom}_{A^e}(A^{n+2}, X)$,

$$f(a_0 \otimes \cdots \otimes a_{n+1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} f(a_0 \otimes \cdots \otimes a_{n+1}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in X_{AA},$$

the canonical inclusion $i_n: \mathrm{Hom}_{A^e}(A^{n+2}, X_{AA}) \rightarrow \mathrm{Hom}_{A^e}(A^{n+2}, X)$ is an isomorphism. Let $\theta_n^A: \mathrm{Hom}_{A^e}(A^{n+2}, X) \rightarrow \mathrm{Hom}_{E^e}(X_n^A, X)$ be the map defined by

$$\begin{aligned}\theta_n^A(f)(a_0 \otimes \cdots \otimes a_{n+1}) &= f(a_0 \otimes \cdots \otimes a_{n+1}) && \text{for } a_i \in A, \\ \theta_n^A(f)(a_0 \otimes \cdots \otimes a_n \otimes m) &= f(a_0 \otimes \cdots \otimes a_n \otimes 1_A) \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} && \text{for } a_i \in A \text{ and } m \in M,\end{aligned}$$

and let $\vartheta_n^A: \mathrm{Hom}_{E^e}(X_n^A, X) \rightarrow \mathrm{Hom}_{A^e}(A^{n+2}, X)$ be the map defined by restriction. Clearly $\vartheta_n^A \circ \theta_n^A = \mathrm{id}$. Let us see that $\theta_n^A \circ \vartheta_n^A = \mathrm{id}$. Let $\varphi \in \mathrm{Hom}_{E^e}(X_n^A, X)$. It is clear that $\theta_n^A \circ \vartheta_n^A(\varphi)(a_0 \otimes \cdots \otimes a_{n+1}) = \varphi(a_0 \otimes \cdots \otimes a_{n+1})$ for all $a_0, \dots, a_{n+1} \in A$. Since

$$\begin{aligned}\varphi(a_0 \otimes \cdots \otimes a_n \otimes m) &= \varphi(a_0 \otimes \cdots \otimes a_n \otimes 1_A) \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \\ &= \theta_n^A(\vartheta_n^A(\varphi))(a_0 \otimes \cdots \otimes a_n \otimes 1_A) \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \\ &= \theta_n^A(\vartheta_n^A(\varphi))(a_0 \otimes \cdots \otimes a_n \otimes m),\end{aligned}$$

for all $a_0, \dots, a_n \in A$ and $m \in M$, we have that $\theta_n^A \circ \vartheta_n^A(\varphi) = \varphi$. As the family $\theta_* \circ i_*$ is a map of complexes, the first assertion holds. The proof of the second one is similar. \square

Lemma 5. *We have*

$$\mathrm{Hom}_{A \otimes B^{\mathrm{op}}}((\frac{X_*}{X_*^A \oplus X_*^B}, b'_*), X_{AB}) \simeq \mathrm{Hom}_{E^e}((\frac{X_*}{X_*^A \oplus X_*^B}, b'_*), X).$$

Proof. Since, for every $f \in \mathrm{Hom}_{E^e}(\frac{X_n}{X_n^A \oplus X_n^B}, X)$,

$$f(x_0 \otimes \cdots \otimes x_{n+1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} f(x_0 \otimes \cdots \otimes x_{n+1}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in X_{AB},$$

the canonical inclusion $\mathrm{Hom}_{E^e}(\frac{X_n}{X_n^A \oplus X_n^B}, X_{AB}) \rightarrow \mathrm{Hom}_{E^e}(\frac{X_n}{X_n^A \oplus X_n^B}, X)$ is an isomorphism. To end the proof it suffices to observe that

$$\mathrm{Hom}_{E^e}(\frac{X_n}{X_n^A \oplus X_n^B}, X_{AB}) \simeq \mathrm{Hom}_{A \otimes B^{\mathrm{op}}}(\frac{X_n}{X_n^A \oplus X_n^B}, X_{AB}). \quad \square$$

Proof of Theorem 1. Because of Lemma 3, the short exact sequence

$$0 \rightarrow (X_*^A, b'_*) \oplus (X_*^B, b'_*) \rightarrow (X_*, b'_*) \rightarrow \left(\frac{X_*}{X_*^A \oplus X_*^B}, b'_* \right) \rightarrow 0,$$

gives rise to the long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ext}_{E^e, k}^0(E, X) &\rightarrow \text{Ext}_{E^e, k}^0(1_A E \oplus E 1_B, X) \rightarrow \text{Ext}_{E^e, k}^0(M, X) \rightarrow \\ &\rightarrow \text{Ext}_{E^e, k}^1(E, X) \rightarrow \text{Ext}_{E^e, k}^1(1_A E \oplus E 1_B, X) \rightarrow \text{Ext}_{E^e, k}^1(M, X) \rightarrow \dots \end{aligned}$$

To end the proof it suffices to apply Lemmas 4 and 5. \square

Lemma 6. Let $\mu: \frac{X_0}{X_0^A} \rightarrow B_E$ be the map defined by $\mu(b_0 \otimes b_1 + m \otimes b) = b_0 b_1$, for $b, b_0, b_1 \in B$ and $m \in M$. The complex

$$(*) \quad B_E \xleftarrow{\mu} \frac{X_0}{X_0^A} \xleftarrow{b'_1} \frac{X_1}{X_1^A} \xleftarrow{b'_2} \frac{X_2}{X_2^A} \xleftarrow{b'_3} \frac{X_3}{X_3^A} \xleftarrow{b'_4} \frac{X_4}{X_4^A} \xleftarrow{b'_5} \frac{X_5}{X_5^A} \xleftarrow{b'_6} \dots,$$

is a relative projective resolution of B_E as an E -bimodule. A contracting homotopy of $(*)$ as a complex of k -modules is the family $\sigma_0: B_E \rightarrow \frac{X_0}{X_0^A}$ and $\sigma_{n+1}: \frac{X_n}{X_n^A} \rightarrow \frac{X_{n+1}}{X_{n+1}^A}$ ($n \geq 0$), defined by:

$$\sigma_{n+1}(x_0 \otimes \dots \otimes x_n) = 1_A \otimes x_0 \otimes \dots \otimes x_n.$$

Proof. It follows by a direct computation. \square

Lemma 7. We have $\text{Hom}_{E \otimes B^{\text{op}}}((\frac{X_*}{X_*^A}, b'_*), X 1_B) \simeq \text{Hom}_{E^e}((\frac{X_*}{X_*^A}, b'_*), X)$.

Proof. Since, for every $f \in \text{Hom}_{E^e}(\frac{X_n}{X_n^A}, X)$,

$$f(x_0 \otimes \dots \otimes x_{n+1}) = f(x_0 \otimes \dots \otimes x_{n+1}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in X 1_B,$$

the canonical inclusion $\text{Hom}_{E^e}(\frac{X_n}{X_n^A}, X 1_B) \rightarrow \text{Hom}_{E^e}(\frac{X_n}{X_n^A}, X)$ is an isomorphism. To end the proof it suffices to observe that

$$\text{Hom}_{E \otimes B^{\text{op}}}(\frac{X_n}{X_n^A}, X 1_B) = \text{Hom}_{E^e}(\frac{X_n}{X_n^A}, X 1_B). \quad \square$$

Proof of Theorem 2. Because of Lemma 6, the short exact sequence

$$0 \rightarrow (X_*^A, b'_*) \rightarrow (X_*, b'_*) \rightarrow \left(\frac{X_*}{X_*^A}, b'_* \right) \rightarrow 0,$$

gives rise to the long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ext}_{E^e, k}^0(B_E, X) &\rightarrow \text{Ext}_{E^e, k}^0(E, X) \rightarrow \text{Ext}_{E^e, k}^0(1_A E, X) \rightarrow \\ &\rightarrow \text{Ext}_{E^e, k}^1(B_E, X) \rightarrow \text{Ext}_{E^e, k}^1(E, X) \rightarrow \text{Ext}_{E^e, k}^1(1_A E, X) \rightarrow \dots \end{aligned}$$

To end the proof it suffices to apply Lemmas 4 and 7. \square

Lemma 8. *Let $((A \otimes B^{\text{op}})^{*+1} \otimes M), b'_*$ be the bar resolution of M as a left $A \otimes B^{\text{op}}$ -module. There is a map of resolutions $\gamma_*: ((A \otimes B^{\text{op}})^{*+1} \otimes M), b'_* \rightarrow \left(\frac{X_{*+1}}{X_{*+1}^A \oplus X_{*+1}^B}, b'_{*+1}\right)$, defined by*

$$\gamma_n((a_0 \otimes b_0) \otimes \cdots \otimes (a_n \otimes b_n) \otimes m) = \sum_{i=0}^n (-1)^{\binom{n-i}{2}} \mathbf{a}_{0,i} \otimes \mathbf{a}_{i+1,n} m \otimes \mathbf{b}_{n,i+1} \otimes \mathbf{b}_{i,0},$$

where

$$\begin{aligned} \mathbf{a}_{0,i} &= a_0 \otimes a_1 \otimes \cdots \otimes a_i, & \mathbf{a}_{i+1,n} &= a_{i+1} a_{i+2} \cdots a_n, \\ \mathbf{b}_{n,i+1} &= b_n \otimes b_{n-1} \otimes \cdots \otimes b_{i+1}, & \mathbf{b}_{i,0} &= b_i b_{i-1} \cdots b_0. \end{aligned}$$

Proof. We must prove that $\mu \circ \gamma_0 = b'_0$, where μ is the map introduced in Lemma 3 and that $\gamma_{n-1} \circ b'_n = b'_n \circ \gamma_n$ for all $n \geq 1$. The first assertion is evident. Let us check the second one. We have

$$\begin{aligned} & \gamma_{n-1} \circ b'_n((a_0 \otimes b_0) \otimes \cdots \otimes (a_n \otimes b_n) \otimes m) \\ &= \gamma_{n-1} \left(\sum_{j=0}^{n-1} (-1)^j (a_0 \otimes b_0) \otimes \cdots \otimes (a_j a_{j+1} \otimes b_{j+1} b_j) \otimes \cdots \otimes (a_n \otimes b_n) \otimes m \right) \\ &+ (-1)^n \gamma_{n-1}((a_0 \otimes b_0) \otimes \cdots \otimes (a_{n-1} \otimes b_{n-1}) \otimes a_n m b_n) \\ &= \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{j+\binom{n-i-1}{2}} \mathbf{a}_{0,j-1} \otimes a_j a_{j+1} \otimes \mathbf{a}_{j+2,i+1} \otimes \mathbf{a}_{i+2,n} m \otimes \mathbf{b}_{n,i+2} \otimes \mathbf{b}_{i+1,0} \\ &+ \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} (-1)^{j+\binom{n-i-1}{2}} \mathbf{a}_{0,i} \otimes \mathbf{a}_{i+1,n} m \otimes \mathbf{b}_{n,j+2} \otimes b_{j+1} b_j \otimes \mathbf{b}_{j-1,i-1} \otimes \mathbf{b}_{i,0} \\ &+ \sum_{i=0}^{n-1} (-1)^{n+\binom{n-i-1}{2}} \mathbf{a}_{0,i} \otimes \mathbf{a}_{i+1,n} m b_n \otimes \mathbf{b}_{n-1,i+1} \otimes \mathbf{b}_{i,0} \\ &= \sum_{i=1}^n \sum_{j=0}^{i-1} (-1)^{\binom{n-i}{2}+j} \mathbf{a}_{0,j-1} \otimes a_j a_{j+1} \otimes \mathbf{a}_{j+2,i} \otimes \mathbf{a}_{i+1,n} m \otimes \mathbf{b}_{n,i+1} \otimes \mathbf{b}_{i,0} \\ &+ \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} (-1)^{\binom{n-i}{2}+i+1+n-j} \mathbf{a}_{0,i} \otimes \mathbf{a}_{i+1,n} m \otimes \mathbf{b}_{n,j+2} \otimes b_{j+1} b_j \otimes \mathbf{b}_{j-1,i} \otimes \mathbf{b}_{i-1,0} \\ &+ \sum_{i=0}^{n-1} (-1)^{\binom{n-i}{2}+i+1} \mathbf{a}_{0,i} \otimes \mathbf{a}_{i+1,n} m b_n \otimes \mathbf{b}_{n-1,i+1} \otimes \mathbf{b}_{i,0} \\ &= b'_n \left(\sum_{i=0}^n (-1)^{\binom{n-i}{2}} \mathbf{a}_{0,i} \otimes \mathbf{a}_{i+1,n} m \otimes \mathbf{b}_{n,i+1} \otimes \mathbf{b}_{i,0} \right) \\ &= b'_n \circ \gamma_n((a_0 \otimes b_0) \otimes \cdots \otimes (a_n \otimes b_n) \otimes m). \quad \square \end{aligned}$$

Proof of Theorem 3. 1) It is easy to see that $H^*(E, E) \xrightarrow{j^*} H^*(A, A) \oplus H^*(B, B)$ is induced by the canonical restriction

$$\mathrm{Hom}_{E^e}((E^{*+2}, b'_*), E) \rightarrow \mathrm{Hom}_{A^e}((A^{*+2}, b'_*), A) \oplus \mathrm{Hom}_{B^e}((B^{*+2}, b'_*), B).$$

From this fact follows immediately that j^* is a map of graded rings.

2) We prove the second assertion. The first one follows similarly. It is easy to see that $H^*(B, B) \rightarrow \mathrm{Ext}_{A \otimes B^{\mathrm{op}}, k}^*(M, M)$ is induced by the map of complexes

$$\mathrm{Hom}_{B^e}((B^{*+2}, b'_*), B) \xrightarrow{\tilde{\delta}^*} \mathrm{Hom}_{A \otimes B^{\mathrm{op}}} \left(\left(\frac{X_{*+1}}{X_{*+1}^A \oplus X_{*+1}^B}, -b'_{*+1} \right), M \right),$$

defined by

$$\tilde{\delta}^n(f)(\mathbf{a}_{0,i} \otimes m \otimes \mathbf{b}_{i+1,n+1}) = \begin{cases} (-1)^n f(\mathbf{a}_{0,n} \otimes 1_A) m b_{n+1} & \text{if } i = n, \\ 0 & \text{in other cases,} \end{cases}$$

where $f \in \mathrm{Hom}_{B^e}(B^{n+2}, B)$, $\mathbf{a}_{0,i} = a_0 \otimes \cdots \otimes a_i$ and $\mathbf{b}_{i+1,n+1} = b_{i+1} \otimes \cdots \otimes b_{n+1}$. Let us consider the morphism

$$\mathrm{Hom}_{A \otimes B^{\mathrm{op}}} \left(\left(\frac{X_{*+1}}{X_{*+1}^A \oplus X_{*+1}^B}, b'_{*+1} \right), M \right) \xrightarrow{\tilde{\gamma}^*} \mathrm{Hom}_{A \otimes B^{\mathrm{op}}} \left(((A \otimes B^{\mathrm{op}})^{*+1} \otimes M), b'_*, M \right)$$

induced by the map γ_* of Lemma 8. Let $\phi^* = \tilde{\gamma}^* \circ \tilde{\delta}^*$. It is immediate that

$$\phi^n(f)((a_0 \otimes b_0) \otimes \cdots \otimes (a_n \otimes b_n) \otimes m) = (-1)^{\binom{n}{2}} a_0 \cdots a_n m f(1_B \otimes b_n \otimes \cdots \otimes b_0).$$

Hence, for $f \in \mathrm{Hom}_{B^e}(B^{r+2}, B)$, $g \in \mathrm{Hom}_{B^e}(B^{s+2}, B)$ and $x_0 \otimes \cdots \otimes x_{r+s} = (a_0 \otimes b_0) \otimes \cdots \otimes (a_{r+s} \otimes b_{r+s})$, we have

$$\begin{aligned} & \phi^r(f) \phi^s(g)(x_0 \otimes \cdots \otimes x_{r+s} \otimes m) \\ &= \phi^r(f)(x_0 \otimes \cdots \otimes x_r \otimes \phi^s(g)((1_A \otimes 1_B) \otimes x_{r+1} \otimes \cdots \otimes x_{r+s} \otimes m)) \\ &= (-1)^{\binom{s}{2}} \phi^r(f)(x_0 \otimes \cdots \otimes x_r \otimes a_{r+1} \cdots a_{r+s} m g(1_B \otimes b_{r+s} \otimes \cdots \otimes b_{r+1} \otimes 1_B)) \\ &= (-1)^{rs + \binom{r+s}{2}} a_0 \cdots a_{r+s} m g(1_B \otimes b_{r+s} \otimes \cdots \otimes b_{r+1} \otimes 1_B) f(1_B \otimes b_r \otimes \cdots \otimes b_0) \\ &= (-1)^{rs} (-1)^{\binom{r+s}{2}} a_0 \cdots a_{r+s} m (gf)(1_B \otimes b_{r+s} \otimes \cdots \otimes b_0) \\ &= \phi^{r+s}((-1)^{rs} gf)(x_0 \otimes \cdots \otimes x_{r+s} \otimes m). \end{aligned}$$

This finished the proof, since $H^*(B, B)$ is graded commutative.

3) The complex $\mathrm{Hom}_{E^e}((X_*, b'_*), E)$ is a differential graded algebra with the product defined by $(fg)(x_0 \otimes \cdots \otimes x_{r+s+1}) = f(x_0 \otimes \cdots \otimes x_r \otimes 1_E) g(1_E \otimes x_{r+1} \otimes \cdots \otimes x_{r+s+1})$, for $f \in \mathrm{Hom}_{E^e}(X_r, E)$ and $g \in \mathrm{Hom}_{E^e}(X_s, E)$. It is immediate that this product induces the cup product in $H^*(E, E)$. Assume that $s \geq 1$ and that f belongs to the image of $\mathrm{Hom}_{E^e} \left(\frac{X_r}{X_r^A \oplus X_r^B}, E \right) \rightarrow \mathrm{Hom}_{E^e}(X_r, E)$. Let $g' \in \mathrm{Hom}_{E^e}(X_s, E)$ the cocycle defined by

$$g'(x_0 \otimes \cdots \otimes x_r) = \begin{cases} g(x_0 \otimes \cdots \otimes x_r) & \text{if } x_0 \otimes \cdots \otimes x_r \in A^r \otimes (A \oplus M) \\ 0 & \text{in other case} \end{cases}$$

It is easy to check that $gf = g'f$ and that $fg' = 0$. Since $g'f$ is homologous to $(-1)^{rs} fg'$ we have that the class of gf in $H^{r+s}(E, E)$ is zero. \square

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JORGE ALBERTO GUCCIONE, DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, PABELLÓN 1 - CIUDAD UNIVERSITARIA, (1428) BUENOS AIRES, ARGENTINA.

E-mail address: `vander@dm.uba.ar`

JUAN JOSÉ GUCCIONE, DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, PABELLÓN 1 - CIUDAD UNIVERSITARIA, (1428) BUENOS AIRES, ARGENTINA.

E-mail address: `jggucci@dm.uba.ar`